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Convex circuit-free coloration of an oriented graph[☆]

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ABSTRACT

We introduce the *Convex Circuit-Free coloration* and *Convex Circuit-Free chromatic number* $\overrightarrow{\chi_a}(\overrightarrow{G})$ of an oriented graph $\overrightarrow{G}$ and establish various basic results. We show that the problem of deciding if an oriented graph verifies $\chi_a(\overrightarrow{G}) \leq k$ is NP-complete if $k \geq 3$, and polynomial if $k \leq 2$. We introduce an algorithm which finds the optimal convex circuit-free coloration for tournaments, and characterize the tournaments that are *vertex-critical* for the convex circuit-free coloration.

1. Introduction

A convex subset of a tournament is a vertex subset with the property that every 2-directed path beginning and ending inside the convex subset is completely contained within the subset. In this paper, we investigate the coloration of an oriented graph $\overrightarrow{G}$ into circuit free subsets which are convex subsets of a tournament defined on $V(\overrightarrow{G})$ and containing $\overrightarrow{G}$ as a subgraph (note that the tournament must be the same for all these convex subsets). Such a coloration of an oriented graph $\overrightarrow{G}$ is referenced in the following by *CCF-coloration* for 'Convex Circuit-Free coloration'. If we color each subset with a different color, such a coloration appears as an extension of the notion of oriented coloring introduced by Sopena [11]. Indeed, within an oriented coloring, each monochromatic subgraph is not only without circuit but also without arc (independent set). In the same way, as an oriented coloration may be defined by means of oriented homomorphism [6,11], the CCF-coloration may be equivalently defined by the notion of *circuit-free homomorphism* (called *acyclic homomorphism* in [4]). A *circuit-free homomorphism* of a digraph $\overrightarrow{G}$ into a digraph $\overrightarrow{F}$ is a mapping ϕ from $V(\overrightarrow{G})$ to $V(\overrightarrow{F})$ such that:

(i) for every arc $(u, v) \in A(\overrightarrow{G})$, either $\phi(u) = \phi(v)$ or $(\phi(u), \phi(v))$ is an arc of $\overrightarrow{F}$,
(ii) for every vertex $v \in V(\overrightarrow{F})$, the induced oriented graph $\overrightarrow{G}(\phi^{-1}(v))$ is circuit-free.

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An oriented graph $\overrightarrow{G}$ admits a k-CCF coloration if and only if there exists an oriented graph $\overrightarrow{F}$ of order k and a circuit-free homomorphism of $\overrightarrow{G}$ into $\overrightarrow{F}$. Such a minimal k is called *CCF-chromatic number* of $\overrightarrow{G}$ and denoted by $\overrightarrow{\chi_a}(\overrightarrow{G})$. That type of coloration was originally motivated by the search of structures in large majority voting tournaments ([8]).

Let us give some notations and definitions. All digraphs considered here are finite and have no loop or multiple edge. A circuit is a directed cycle. An oriented graph is a digraph without circuit of length two. In other words, it is an orientation of a simple graph. An oriented graph T is a tournament if and only if it is complete, *i.e.* for every pair $\{i, j\}$ of vertices, (i, j) or (j, i) is an arc of T. The *dual tournament* of T is obtained by reversing the arcs of T. Finally, for a graph having property $\mathcal{P}$ we say that G is vertex-critical for $\mathcal{P}$ if it loses the property $\mathcal{P}$ whenever any vertex is removed. The set of vertices and the set of arcs of a digraph $\overrightarrow{G}$ are respectively denoted by $V(\overrightarrow{G})$ and $A(\overrightarrow{G})$. If (x, y) is an arc of $\overrightarrow{G}$, then we say that x dominates y or y is a successor of x and that x is a predecessor of y. We shall use the notation $x \to y$ to denote this. We respectively denote by $\Gamma^+(x)$ and $\Gamma^-(x)$ the set of successors and the set of predecessors of x. The in-degree of a vertex x is the cardinal of $\Gamma^-(x)$, and the out-degree of x is the cardinal of $\Gamma^+(x)$. If A and B are disjoint subsets of $V(\overrightarrow{G})$ such that all arcs between A and B are directed toward B, then we use the notation $A \to B$ and say that the sets A and B verify the *unidirection property* or are *in unidirection*. The absence of arcs between A and B is a particular case of unidirection. Previous definitions of a *CCF-coloration* are both equivalent to a partition of the vertex set into circuit free subsets with unidirection property between any two of them. For a subset B of $V(\overrightarrow{G})$, $\overrightarrow{G} \setminus B$ denotes the subdigraph of $\overrightarrow{G}$ obtained after removing the vertices of B and all the arcs with at least one extremity in B. The subdigraph induced by a vertex subset B of $\overrightarrow{G}$ is defined as $\overrightarrow{G} \setminus \left(V(\overrightarrow{G}) \setminus B\right)$ and is denoted by $\overrightarrow{G}\,(B)$.

The paper is organized in two parts. In the first one, we prove that the minimization problem of finding the smallest integer k such that G has a CCF-coloration in k classes is of polynomial complexity if G is a tournament and NP-complet in the general case. In a second part we focus on the CCF-indecomposable tournaments, that is tournaments T for which $\overrightarrow{\chi_a(T)}$ is equal to the number of vertices. That class is large since the probability that a tournament belongs to it tends toward one when the number of its vertices goes to the infinity. Here, we characterize tournaments that are CCF-indecomposable and critical for that property.

Questions related to the minimum subsets of a CCF-coloration are also closed in their formulation to those of the dichromatic number [2]. The dichromatic number is calculated to avoid monochromatic circuits when a CCF-coloration is caraterized by the absence of dichromatic circuits. In fact the CCF-coloration may be seen as the satisfaction of two properties on the subsets: circuit-free and convexity. In the particular case of tournaments, both of these properties have been studied separately by several authors. In the case of tournaments circuit-free subsets are the transitive ones and [9] characterizes some critically r-dichromatic tournaments. Such tournaments have a partition of its vertex set in at least r transitive subsets and are critical for that property. In the case of tournaments, convex subsets are also called *intervals* [7] or *modules*, [12] and transitive convex subsets are also called *clan*, [1]. The critically indecomposable tournaments are characterized by Schmerl and Trotter in [10]. Indecomposable tournaments (that are tournaments which convex subsets are the singletons, the empty set and the whole vertex set) are CCF-indecomposable. A structural theorem on indecomposable graphs is provided in [7]. The CCF-indecomposable tournaments, also called *primitive tournaments* in [1], are the tournaments without non trivial clan (the trivial clans are $\emptyset$ or $\{x\}$ where $x \in V(T)$). Let us notice that if a tournament admits a non trivial clan then it admits a clan of size 2.

2. Complexity of the CCF-chromatic number problem

For the oriented chromatic number, the threshold between the "easy" and the "hard" computable oriented chromatic number is between 3 and 4. For the CCF-coloration, deciding whether the CCF-chromatic number is less or equal to 3 is already NP-complete.

Let k be a fixed positive integer. The k-**CCF Col** problem is the following decision problem:

k-**CCF Col** (CCF-chromatic number $\leq k$).
Instance: An oriented graph $\overrightarrow{G}$.
Question: Does $\overrightarrow{G}$ admit a k-CCF coloration?

We first note that an oriented graph $\overrightarrow{G}$ admits a 1-CCF coloration if and only if $\overrightarrow{G}$ is circuit-free. Moreover, if $\overrightarrow{G}$ admits a 2-CCF coloration then $\overrightarrow{G}$ is circuit-free and admits a 1-CCF coloration. Hence 1**-CCF Col** and 2**-CCF Col** can be solved in polynomial time.

Theorem 1. *The decision problem* 3***-CCF Col*** *is NP-complete, even if the input is restricted to connected oriented graphs.*

Proof. It is clear that the 3**-CCF Col** problem belongs to *NP*. To show its NP-completness, we shall describe a polynomial-time reduction from 3**-Sat** to 3**-CCF Col**.

Let us consider an instance $(X, \mathcal{C})$ of 3**-Sat**, where $X = \{x_1, x_2, \ldots, x_n\}$ is a set of boolean variables and $\mathcal{C} = \{C_1, C_2, \ldots, C_m\}$ contains m clauses of 3 literals (the set of literals is denoted by $\mathcal{L} = \bigcup_{1\leq i\leq n}\{x_i, \overline{x_i}\}$). The clause C_j is denoted by $z_1^j \vee z_2^j \vee z_3^j$, where $\{z_1^j, z_2^j, z_3^j\} \subset \mathcal{L}$. Since we may assume that no clause is a tautology (*i.e.* contains x_i and $\overline{x_i}$), we will consider that the indexes of literals of any clause are strictly increasing.

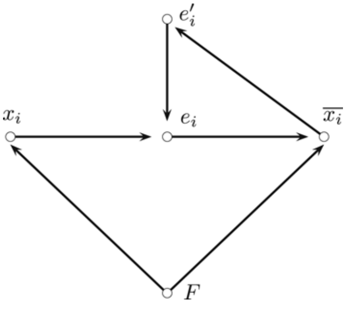
To such an instance of 3**-Sat**, we associate the following oriented graph $\overrightarrow{G}$:

$$V(\overrightarrow{G}) = \bigcup_{1\leq i\leq n}\{x_i, e_i, e_i', \overline{x_i}\} \cup \bigcup_{1\leq j\leq m}\{c_1^j, c_2^j, c_3^j, c_4^j, c_5^j, c_6^j, F_1^j, F_2^j, F_3^j\} \cup \{T, F, I\}.$$

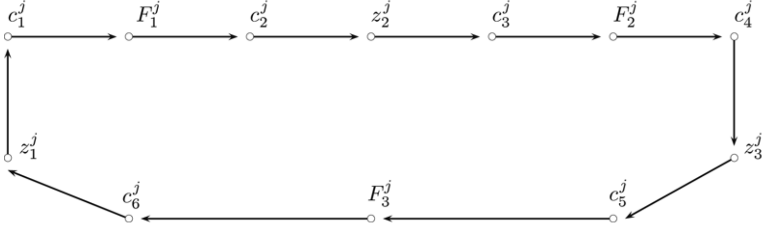
The arc set of $\overrightarrow{G}$ is the union of four types of arcs:

First type: For all integer $i \in \{1, 2, \ldots, n\}$, we have the set of arcs

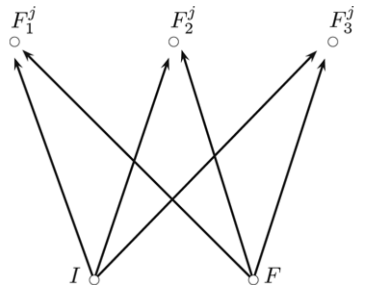
$$\{(e_i', e_i), (e_i, \overline{x_i}), (\overline{x_i}, e_i'), (x_i, e_i), (F, x_i), (F, \overline{x_i})\}.$$



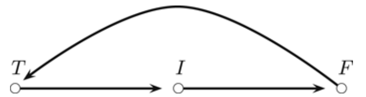
Second type: For all $j \in \{1, 2, \ldots, m\}$, for $C_j = z_1^j \vee z_2^j \vee z_3^j$, we get a copy of the oriented graph $\overrightarrow{K_j}$, identifying the vertices z_1^j, z_2^j, z_3^j to vertices in $\bigcup_{1\leq i\leq n}\{x_i, \overline{x_i}\}$:



Third type: For all $j \in \{1, 2, \ldots, m\}$, we have: $\{I, F\} \to \{F_1^j, F_2^j, F_3^j\}$. Then, we obtain a copy of the following oriented graph:



Fourth type: The induced oriented graph $\overrightarrow{G}(\{V, F, I\})$ is isomorphic to:



The construction of $\overrightarrow{G}$ may be carried out in polynomial time. We claim that $\overrightarrow{G}$ is 3 CCF-decomposable if and only if the clauses $C_1, C_2, \ldots, C_m$ are simultaneously satisfiable.

Let us suppose that the oriented graph $\overrightarrow{G}$ admits a 3-CCF-coloration. The arcs of the fourth type imply that there exists a circuit-free homomorphism ϕ from $\overrightarrow{G}$ to the 3-circuit $(1, 2, 3)$. Without loss of generality, we may assume that $\phi(T) = 1$, $\phi(I) = 2$ and $\phi(F) = 3$. The arcs of the first type imply that, for each i in $\{1, 2, \ldots, n\}$, $\{\phi(x_i), \phi(\overline{x_i})\} = \{1, 3\}$. Since the vertices $\{F_l^j\}_{\substack{1\le j\le m\\ 1\le l\le 3}}$ are successors of I and F, then $\forall j \in \{1, \ldots, m\}$, $\forall l \in \{1, 2, 3\}$, $\phi(F_l^j) = 3$. Given an integer $j \in \{1, 2, \ldots, m\}$, let us suppose that $\phi(z_1^j) = \phi(z_2^j) = \phi(z_3^j) = 3$, then, for all $l \in \{1, 2, \ldots, 6\}$, $\phi(c_l^j) = 3$. Then, $K_j \subset \phi^{-1}(3)$, which contradicts the fact that ϕ is a circuit-free homomorphism.

Then, at least one of the vertices $\{z_1^j, z_2^j, z_3^j\}$ is in the monochromatic class $\phi^{-1}(1)$. The truth distribution $\mathcal{T} : X \to \{\texttt{True}, \texttt{False}\}$ defined by

$$\begin{cases} \mathcal{T}(x_i) = \texttt{True} & \text{if } \phi(x_i) = 1, \\ \mathcal{T}(x_i) = \texttt{False} & \text{if } \phi(x_i) = 3 \end{cases}$$

satisfies all the clauses $\{C_j\}_{1\le j\le m}$ of the 3-**Sat** instance.

Conversely, suppose that $\mathcal{T} : X \to \{\texttt{True}, \texttt{False}\}$ is a satisfying truth assignment for the clauses $C_1, C_2, \ldots, C_m$. Then, we define the circuit-free homomorphism ϕ from $V(\overrightarrow{G})$ into the set of vertices of the 3-circuit $(1, 2, 3)$ by $\phi(T) = 1$, $\phi(I) = 2$ and $\phi(F) = 3$.

$$\begin{cases} \text{if } \phi(x_i) = \texttt{True} & \text{then } \phi(x_i) = 1 \quad \text{and} \quad \phi(\overline{x_i}) = 3; \\ & \text{else } \phi(x_i) = 3 \quad \text{and} \quad \phi(\overline{x_i}) = 1. \end{cases}$$

For every integer $j \in \{1, 2, \ldots, m\}$, $\phi(F_1^j) = \phi(F_2^j) = \phi(F_3^j) = 3$, and, for $k \in \{1, 2, 3\}$, if $\phi(z_k^j) = 3$ then $\phi(c_k^j) = 3$ else $\phi(c_k^j) = 1$.

Such a mapping is a 3 circuit-free homomorphism from $\overrightarrow{G}$ to the 3-circuit $(1, 2, 3)$, and then $\overrightarrow{G}$ admits a 3-CCF coloration. $\square$

3. The case of tournaments

In this section, we investigate the complexity of the k-**CCF Col** problem over the family of tournaments. Let $T = (V, A)$ be a tournament and x a vertex of T. If it exists, we define and denote by x^+ the *highest successor* of x as the vertex of $\Gamma^+(x)$ which verifies the equality $\Gamma^+(x^+) = \Gamma^+(x) \setminus \{x^+\}$. Given a tournament T and a vertex x, we can compute x^+ in polynomial time by the following greedy algorithm:

Down(x)

Input: A tournament T and a vertex x of T.

Output: A vertex y such that $y = x^+$ if it exists, $\emptyset$ if not.

We denote by $\{y_1, y_2, \dots, y_k\}$ the set $\Gamma^+(x)$, $i = 1$ and $x^+ = \emptyset$.
While $i \le k$ Do:
If y_i verifies $\Gamma^+(y_i) = \Gamma^+(x) \setminus \{y_i\}$, then $x^+ = y_i$ and $i = k + 1$;
Else $i = i + 1$.
Return(x^+).

Proposition 1. *Let T be a tournament of order n with $\overrightarrow{\chi_a}(T) = k$ and x a vertex of T.*

(i) *If there is a k-CCF coloration c of T such that x is not the smallest vertex in its monochromatic class then x^+ exists.*

(ii) *Conversely, if x^+ exists then for all convex circuit-free k-coloration c of T, $c(x) = c(x^+)$.*

Proof. (i) Let $C_x = c^{-1}(c(x))$ be the CCF-monochromatic class of x, and let us suppose that x is not the smallest vertex of C_x. The intersection of the induced subdigraphs $\overrightarrow{G}(C_x)$ and $\overrightarrow{G}(\Gamma^+(x))$ is a non-empty order. Let y be the highest vertex of this order, since any other CCF-monochromatic class is in unidirection with C_x, we have $\Gamma^+(y) = \Gamma^+(x) \setminus \{y\}$. Then y is the highest successor of x and $y = x^+$.

(ii) Let c be a k-CCF coloration of T and let us suppose, for contradiction, that $y = x^+$ and $c(x) \neq c(y)$. We denote by C_1 and C_2 the color classes of x and y respectively, and by $\{C_j\}_{3\le j\le k}$ the other CCF-monochromatic classes of T. We have $x \to y$ and by convexity $C_1 \to C_2$. Moreover, since $y = x^+$, we have: $\forall j \in \{3, 4, \dots, k\}$, $[C_1 \to C_j] \Leftrightarrow [C_2 \to C_j]$. Consequently, the $(k - 1)$-partition $\{C_1 \cup C_2, C_3, \dots, C_k\}$ is a $(k-1)$-CCF-coloration of T, which contradicts the equality $\overrightarrow{\chi_a}(T) = k$. □

Previous results are also true if we consider predecessors instead of successors. If it exists, we define the *smallest predecessor* of x as, the vertex of $\Gamma^-(x)$ such that $\Gamma^-(x^-) = \Gamma^-(x) \setminus \{x^-\}$. It could be computed in polynomial time by the following greedy algorithm:

Up(x)

Input: A tournament T and a vertex x of T.

Output: A vertex y such that $y = x^-$ if it exists, $\emptyset$ if not.

We denote by $\{y_1, y_2, \dots, y_k\}$ the set $\Gamma^-(x)$, $i = 1$ and $x^- = \emptyset$.
While $i \le k$ Do:
If y_i verifies $\Gamma^-(y_i) = \Gamma^-(x) \setminus \{y_i\}$, then $x^- = y_i$ and $i = k + 1$;
Else $i = i + 1$.
Return(x^-).

Proposition 2. *Let T be a tournament with $\overrightarrow{\chi_a}(T) = k$ and let x be a vertex of T.*

(i) *If it exists a k-CCF coloration such that x does not dominate all vertices of its CCF-monochromatic class then x^- exists.*

(ii) *Conversely, if x^- exists then for every k-CCF coloration c of T, $c(x) = c(x^-)$.*

Corollary 1. *Let T be a tournament such that $\overrightarrow{\chi_a}(T) = k$. The k-CCF coloration of T is unique.*

Proof. Let c be a k-CCF-coloration of T. We then have the following equivalence:

$$\left[c(x) = c(y) \text{ and } y \text{ is the direct successor of } x \text{ whithin the order } c^{-1}(c(x))\right] \Leftrightarrow y = x^+.$$

Then, since the highest successor and the smallest predecessor are unique (if there exist), we deduce the unicity of the optimal convex circuit-free coloration. □

The following algorithm `OptDec` computes in polynomial time the optimal CCF-coloration of a tournament T.

Algorithm OptDec
  Input: A tournament T
  Output: The optimal CCF-coloration of T

For every vertex $x \in V(T)$, let $\mathcal{M}(x)$ denote a mark.
Initialization: $\forall x \in V(T)$, $\mathcal{M}(x) = 0$ and $k = 0$.
While a vertex x such that $\mathcal{M}(x) = 0$ exists, DO:
    $k \leftarrow k + 1$
    $v \leftarrow x$
    $\mathcal{M}(x) \leftarrow k$
  While $Up(v) \neq \emptyset$, DO:
    $v \leftarrow Up(v)$
    $\mathcal{M}(v) = k$ end.
    $v \leftarrow x$
  While $Down(v) \neq \emptyset$, DO:
    $v \leftarrow Down(v)$,
    $\mathcal{M}(v) = k$ end.
End.

Proposition 3. *Given a tournament T with $\overrightarrow{\chi_a}(T) = k$, the optimal $CCF - k$ coloration is computed in polynomial time by the algorithm* `OptDec`*. The optimal CCF-coloration c of T is given by $\forall x \in V(T)$, $c(x) = \mathcal{M}(x)$.*

The previous algorithm provides a partition of $V(T)$ into maximal clans under inclusion that are the CCF-monochromatic classes. Let us recall the definition of the quotient of a tournament by a convex partition. A partition P of $V(T)$ is a *convex partition* (or *interval partition*) of T when each element of P is a convex subset of T. For such a partition P, the *quotient* T/P of T by P is the tournament defined on $V(T/P) = P$ as follows: given $X \neq Y \in P$, (X, Y) is an arc of T/P if $X \to Y$ in T. We now associate with T the family $\Pi(T)$ of the maximal clans of T which is an interval partition of T. We have $\overrightarrow{\chi_a}(T) = |\Pi(T)|$ and we can formulate the results by:

Proposition 4. *For every tournament T with $|V(T)| \geq 2$, one of the following is satisfied:*

(i) *$|\Pi(T)| = 1$ and T is a total order*
(ii) *$|\Pi(T)| \geq 3$ and $T/\Pi(T)$ is CCF-indecomposable (or primitive).*

4. CCF-Indecomposable tournament

The aim of this part is to characterize the vertex-critical CCF-indecomposable tournaments. A n-tournament is *CCF-indecomposable* if $\overrightarrow{\chi_a}(T) = n$. In other words, any convex subset of T with at least 2 vertices contains a circuit. Remark that such an indecomposable oriented graph does not contain convex subset of size two. If $\overrightarrow{G}$ is not CCF-indecomposable then $\overrightarrow{G}$ is called CCF-decomposable. For the following probabilistic proof, we need the notion of *random tournament*, constructed by picking uniformly at random and independently the orientation of every edge of the complete graph K_n (i.e. if $\{x, y\}$ is an edge of K_n, $P((x, y) \in A(T)) = P((y, x) \in A(T)) = \frac{1}{2}$). We denote by $\mathcal{T}_n$ the set of such random tournaments with n vertices.

Proposition 5. *The probability for a tournament $T \in \mathcal{T}_n$ to be CCF-indecomposable tends to 1 when $n \to \infty$.*

Proof. Let $\mathcal{A}$ the event "T is CCF-indecomposable". The event A^c is realized when there exist two vertices x and y such that $\forall z \in V(T) \setminus \{x, y\}$, $(x, z) \in A(T) \Leftrightarrow (y, z) \in A(T)$. We then obtain: $P(A) = 1 - P(A^c) \leq 1 - \binom{n}{2}(\frac{1}{2})^{n-2}$, and so $\lim_{n\to\infty} P(A) = 1$. □

We could easily exhibit a family of CCF-indecomposable tournaments. Let us recall that a tournament is regular if the in- and out-degrees of its vertices are equal. Regular tournaments are CCF-indecomposable, otherwise the existence of both vertices x and x^+ implies $d^+(x^+) = d^+(x) - 1$ (where $d^+(x)$ denotes the outdegree of x). The following proposition shows that we can add a vertex to a regular tournament in order to obtain a CCF-indecomposable tournament with an even number of vertices. Let us remind that a vertex is a *source* if it has no predecessor and a *sink* if it has no successor.

Proposition 6. *Let T be a CCF-indecomposable tournament without source or sink.*

- *Tournament T' obtained by adding a source s to T is CCF-indecomposable.*
- *Tournament T'' obtained by adding a sink p to T' is CCF-indecomposable.*
- *Tournament T''' obtained by reversing the arc (s, p) in T'' is CCF-indecomposable.*

Moreover, we also obtain a CCF-indecomposable tournament by the converse operations (deleting a source or a sink from a CCF-indecomposable tournament).

We now characterize the tournaments that are vertex-critical for the *CCF-indecomposable* property. Tournament T is *vertex-critical CCF-indecomposable* if T is CCF-indecomposable and, for every vertex u of T, $T \setminus \{u\}$ is CCF-decomposable. Given such a tournament, for every vertex u, there exists at least one unordered pair of vertices $\{i_u, j_u\}$ which verifies the unidirection property with every set $\{x\}$ for x in $V(T) \setminus \{u, i_u, j_u\}$. Such a pair is said to be *associated* with vertex u, which is denoted by $u \sim \{i_u, j_u\}$. In that case, there exists a unique 2-directed path between the vertices i_u and j_u and it goes through u.

Lemma 1. *Let u be a vertex of a vertex-critical CCF-indecomposable tournament T, and let $\{i_u, j_u\}$ be a pair associated with u.*

a. *It exists $v \in V(T) \setminus \{u, i_u\}$ such that $i_u \sim \{u, v\}$.*
b. *Let u and v be two vertices of a vertex-critical CCF-indecomposable tournament, we have:*

$$u = v \Leftrightarrow \{i_u, j_u\} = \{i_v, j_v\}.$$

Proof. a. We have $u \sim \{i_u, j_u\}$. There exist $z \neq z' \in V(T) \setminus \{i_u\}$ such that $i_u \sim \{z, z'\}$. Of course, if $\{z, z'\} \cap \{u, j_u\} = \emptyset$ then (z, i_u, z') and (z, j_u, z') are two distinct 2-directed paths between z and z', which contradicts the fact that $\{z, z'\}$ is a convex subset of $T \setminus \{i_u\}$. Then $\{z, z'\} \cap \{u, j_u\} \neq \emptyset$. Let us suppose that $u \notin \{z, z'\}$ then $\{i_u, j_u, z\}$ or $\{i_u, j_u, z'\}$ is a clan of $T \setminus \{u\}$. We are going to prove that $T \setminus \{u\}$ cannot contain a clan C with $|C| \geq 3$.

Let $C = \{x_1, x_2, \ldots, x_n\}$ with $n \geq 3$ and $x_i \to x_j$ for all $i < j$. Furthermore, suppose that C is a maximal clan under inclusion of $T \setminus \{u\}$.

Let us remark that since T is CCF-indecomposable then $x_i \to u$ if and only if $u \to x_{i+1}$ for all $i \in \{1, \ldots, n-1\}$.

- If $x_1 \to u$ then $T \setminus \{x_1\}$ is CCF-indecomposable. By the previous remark, $\{x_i, x_j\}$ and $\{x_i, u\}$ are not clans of $T \setminus \{x_1\}$. Now, let $\{\alpha, \beta\}$ be two distinct vertices of $T \setminus (C \cup \{u\})$. Set $\{\alpha, x_i\}$ is not a clan of $T \setminus \{x_1\}$ otherwise $C \cup \{\alpha\}$ is a clan of $T \setminus \{u\}$ which contradicts the maximality argument. Finally, neither $\{\alpha, u\}$ nor $\{\alpha, \beta\}$ are clans of $T \setminus \{x_1\}$ otherwise they are clans of T.
- If $u \to x_n$ then $T \setminus \{x_n\}$ is CCF-indecomposable by applying the previous assertion to the dual of T.
- If $x_n \to u \to x_1$ then $x_{2k} \to u \to x_{2k+1}$ implies that n is even and $|C| \geq 4$. In that case, $T \setminus \{x_1, x_2\}$ is similar to $T \setminus \{x_1\}$ of the first assertion and consequently is CCF-indecomposable. Then the only non trivial clan of $T \setminus \{x_1\}$ contains x_2 which is impossible.

In conclusion, T has no clan of order 3 and then $u \in \{z, z'\}$.

b. We may easily verify that such a proposition is true for tournaments with less than four vertices. Let us now consider that $|V(T)| \geq 5$. Let us suppose that $u \neq v$ and $i_u = i_v, j_u = j_v$. Then, there exist two different paths between i_u and j_u, which contradicts the remark.

Let us suppose now that the pairs $\{i_u, j_u\} \neq \{i_v, j_v\}$ and $u = v$. Assume for instance that $j_u \neq j_v$ and $j_v \to j_u$. Since $\{i_u, j_u\}$ is a convex subset of $T \setminus \{u\}$, we obtain that $j_v \to i_u$. By Lemma 1a., there exists $\alpha \in V(T) \setminus \{u, j_u\}$ such that $j_u \sim \{u, \alpha\}$. Now, if $\alpha = i_v$ then (j_v, j_u, i_v) is a 2-directed path which contradicts $u \sim \{i_v, j_v\}$. Therefore $i_v \notin \{u, j_u, \alpha\}$. Since $j_u \sim \{u, \alpha\}$ and $i_v \to u$, we have $i_v \to \alpha$. Furthermore $j_v \neq \alpha$ because $j_v \to j_u \to \alpha$ and then $j_v \notin \{u, j_u, \alpha\}$. Since $j_u \sim \{u, \alpha\}$ and $u \to j_v$ we have $\alpha \to j_v$. Consequently (i_v, α, j_v) is a 2-directed path which contradicts $u \sim \{i_v, j_v\}$. $\square$

Let T be a vertex-critical CCF-indecomposable tournament of order n. We may insist on the fact that the pair $\{i_u, j_u\}$ associated with $u \in V(T)$ is unique. We define the graph G_T associated with T by: $V(G_T) = V(T)$ and $\{i, j\} \in E(G_T)$ if it exists $u \in V(T)$ such that $u \sim \{i, j\}$.

Lemma 2. *Let T be a vertex-critical CCF-indecomposable tournament of order n and G_T its associated graph. Then, we have the following properties:*

a. *The degree of any vertex of G_T is less or equal to 2.*
b. *Connected components of G_T are cycles (without chord).*
c. *Let u be a vertex of T and $\{i_u, j_u\}$ be the edge of G_T associated with u. We denote by $\mathcal{C}$ the cycle of G_T which contains $\{i_u, j_u\}$. Then, $u \in \mathcal{C}$.*
d. *The cardinal of any cycle of G_T is odd.*

Proof. a. Given $u \in V(T)$, we define the function $f_u : \Gamma_{G_T}(u) \to V(T)$ as follows, where $\Gamma_{G_T}(u)$ denotes the neighbourhood of u in G_T. For each $v \in \Gamma_{G_T}(u)$, $f_u(v)$ is the unique vertex of T such that $\{i_{f_u(v)}, j_{f_u(v)}\} = \{u, v\}$. By definition, f_u is injective. Moreover, it follows from Lemma 1a. that $f_u(v) \in \{i_u, j_u\}$. Therefore, $|\Gamma_{G_T}(u)| \leq 2$.

b. The equivalence of Lemma 1b. implies that $|V(G_T)| = |E(G_T)|$. Such equality implies that G_T contains at least one cycle. As the degree of every vertex of G_T is bounded by 2, it follows that the connected components of G_T are cycles, and that every vertex of G_T belongs to exactly one cycle.

c. We denote by $(a_0 = i_u, a_1 = j_u, a_2, a_3, \ldots, a_k)$ the cycle $\mathcal{C}$ and suppose that $u \notin \mathcal{C}$. Then (i_u, u, j_u) is a 2-directed path in T. We have $u \to \{a_i, a_{i+1}\}$, for $i \in \{1, 2, \ldots, k-1\}$. This implies that (i_u, u, a_k) is a 2-directed path of T which contradicts the unicity of the pair associated with u. Hence $u \in \mathcal{C}$.

d. Let $\mathcal{C} = (x_0, \ldots, x_k)$ be a cycle in G_T. We suppose that $x_0 \sim \{x_l, x_{l+1}\}$. By Lemma 1a., $\{x_0, x_1\}$ is associated with x_l or x_{l+1}.

First case: Suppose that $x_{l+1} \sim \{x_0, x_1\}$. Using Lemma 1a., we iterate the process: from $[x_0 \sim \{x_l, x_{l+1}\}$ and $x_{l+1} \sim \{x_0, x_1\}]$, we obtain $[x_1 \sim \{x_{l+1}, x_{l+2}\}$ and $x_{l+2} \sim \{x_1, x_2\}]$ and $[x_2 \sim \{x_{l+2}, x_{l+3}\}$ and $x_{l+3} \sim \{x_2, x_3\}] \ldots$. As every vertex of $\mathcal{C}$ must be associated with a unique edge of $\mathcal{C}$, the iterated process ends with $[x_l \sim \{x_k, x_0\}$ and $x_k \sim \{x_{l-1}, x_l\}]$. Then, $k = 2l$ is even, and so the cycle is odd.
Second case: Suppose that $x_l \sim \{x_0, x_1\}$. Previous iterated process leads to a contradiction because a vertex must be associated with an edge incident to it, which is impossible. $\square$

For any integer l, the *circular tournament* $\overrightarrow{C_l}$ is the tournament of order $2l + 1$ defined by: $V(\overrightarrow{C_l}) = \{0, 1, 2, \ldots, 2l\}$ and $(i, j) \in A(\overrightarrow{C_l})$ if $1 \leq j - i \leq l$, where $j - i$ is considered modulo $2l + 1$.

Proposition 7. *Let $\mathcal{C}$ be a cycle of G_T of length $2l + 1$. The induced oriented graph $T(V(\mathcal{C}))$ is isomorphic to $\overrightarrow{C_l}$.*

Proof. With notations of the proof of Lemma 2d., we have: $x_0 \sim \{x_l; x_{l+1}\}$ and $x_{l+1} \sim \{x_0, x_1\}$. Iterating the process, it follows that, for $i \in \{0, 1, \ldots, 2l\}$, $x_i \sim \{x_{l+i+1}; x_{l+i}\}$, where the indexes are considered modulo $2l + 1$.

Let us now suppose that $x_l \to x_0 \to x_{l+1}$ (if not, then we simply consider the dual tournament of $T(V(\mathcal{C}))$). Then, $x_{l+1} \sim \{x_0, x_1\}$ implies $x_{l+1} \to x_1$, and since $x_i \sim \{x_{l+i}, x_{l+i+1}\}$, we deduce that $x_l \to \{x_0, \ldots, x_{l-1}\}$ and $\{x_{l+1}, \ldots, x_{2l}\} \to x_l$.

Then, for every interger $i \in \{0, \ldots, 2l\}$, we obtain (the indexes are considered modulo $2l + 1$): $x_i \to x_j$ for $j \in \{i - 1, \ldots, i - l\}$, and $x_j \to x_i$ for $j \in \{i + 1, \ldots, i + l\}$.
We then recognize that the obtained tournament is isomorphic to $\overrightarrow{C_l}$.
Finally, $T(V(\mathcal{C}))$ or its dual is isomorphic to $\overrightarrow{C_l}$. As a circular tournament is self-dual (that is isomorphic to its dual), we deduce that $T(V(\mathcal{C}))$ is isomorphic to $\overrightarrow{C_l}$. $\square$

Let T be a tournament which vertex set is $V(T) = \{1, 2, \ldots, n\}$ and let $T_1, \ldots, T_n$ be tournaments. The composition $T[T_1, \ldots, T_n]$ (or lexicographic sum) is the tournament obtained from T by substituting each vertex i of T by the tournament T_i: if $(i, j) \in A(T)$, then $T_i \to T_j$. Let us remark that $\{T_1, \ldots, T_n\}$ is an interval partition of $T[T_1, \ldots, T_n]$ and then the quotient $T[T_1, \ldots, T_n]/\{T_1, \cdots, T_n\}$ is equal to T. Such a definition allows us to characterize the vertex-critical CCF-tournament.

Theorem 2. *Every vertex-critical CCF-indecomposable tournament is isomorphic to* $T'[\overrightarrow{C_{k_1}}, \overrightarrow{C_{k_2}}, \ldots, \overrightarrow{C_{k_p}}]$ *where* T' *is a tournament of order* p *and where* $(k_1, k_2, \ldots, k_p) \in (\mathbb{N}^*)^p$.

Proof. Let T be a vertex-critical CCF-tournament and G_T the graph associated with T. We denote by p the number of cycles in G_T. For $1 \leq i < j \leq p$, if $\mathcal{C}_i$ and $\mathcal{C}_j$ are two disjoint cycles of G_T then the subtournaments of T induced by the vertices of $\mathcal{C}_i$ and $\mathcal{C}_j$ verify the unidirection property in T. We define the tournament T' by $V(T') = \{1, 2, \ldots, p\}$ and $(i, j) \in A(T')$ if and only if the subtournament induced by $V(\mathcal{C}_i)$ dominates the subtournament induced by $V(\mathcal{C}_j)$.

For every i in $\{1, 2, \ldots, p\}$, k_i is the integer such that the number of vertices of cycle $\mathcal{C}_i$ is $2k_i + 1$, and by Proposition 7, we deduce that T is isomorphic to $T'[\overrightarrow{C}_{k_1}, \overrightarrow{C}_{k_2}, \ldots, \overrightarrow{C}_{k_p}]$.

Conversely, let X be a convex subset of $T'[\overrightarrow{C_{k_1}}, \overrightarrow{C_{k_2}}, \ldots, \overrightarrow{C_{k_p}}]$ with at least two vertices. If every vertex of X belongs to the same $\overrightarrow{C_i}$, then $\overrightarrow{C_i} \subset X$ because circulant tournaments are CCF-indecomposable. If $\{x, y\} \subset X$ such that x belongs to $\overrightarrow{C_i}$ and y belongs to $\overrightarrow{C_j}$ (with $i \neq j$), then $\overrightarrow{C_i} \cup \overrightarrow{C_j} \subset X$. We conclude that X contains at least a circulant tournament and then a circuit, and so such tournaments are CCF-indecomposable. It is easy to see that $T'[\overrightarrow{C_{k_1}}, \overrightarrow{C_{k_2}}, \ldots, \overrightarrow{C_{k_p}}] \setminus \{x\}$ is CCF-decomposable. $\square$

5. Discussion

We have introduced a new decomposition, called CCF-decomposition, of an oriented graph into convex subgraphs without circuit. CCF-decomposition may be translated in terms of coloration or homomorphism, as it is made in a classical way with other decompositions. We defined a CCF-chromatic number associated with that decomposition and proved that its calculus is generally NP-complete. For tournaments however, we construct a polynomial algorithm that finds an optimal CCF-coloration (*i.e.* with a minimum number of colors) and we characterize the vertex-critical tournaments for the CCF-decomposition. As we noted in the introduction, indecomposable tournaments (with the definition of Schmerl and Trotter [10]) are CCF-indecomposable and it is easy to prove that the trace of the vertex-critically CCF-indecomposable tournaments into the indecomposable tournaments are the circular tournaments. In our paper we prove in more that the trace of the vertex-critically CCF-indecomposable tournaments into the decomposable tournaments are the compositions of circular tournaments. Formulating the question as a decomposition problem, we have to indicate another possible demonstration of our Theorem 2 from the Gallai decomposition theorem of tournaments [5, 3]. Let us indicate in the following the main points of that proof which is at least as long as that presented previously in this paper.

Given a tournament T, a subset X of $V(T)$ is a *strong interval* of T provided that X is an interval of T such that for every interval Y of T, we have: if $X \cap Y \neq \emptyset$ then $X \subseteq Y$ or $Y \subseteq X$. The family of the strong intervals of T realizes a partition $P(T)$ of T.

The Gallai decomposition theorem characterizes the corresponding quotient as follows:

Theorem 3 (*Gallai, 1967*). *For every tournament T, with $|V(T)| \geq 2$, one of the assertions below is satisfied.*

(i) *T is not strongly connected, $P(T)$ is the family of the strongly connected components of T and $T/P(T)$ is a total order*

(ii) *T is strongly connected, $|P(T)| \geq 3$ and $T/P(T)$ is indecomposable.*

In the case of a non-strongly connected tournament T, it is easy to establish the following proposition:

Proposition 8. *Let T be a non strongly connected tournament with $|V(T)| \geq 3$, T is critically CCF-indecomposable if and only if for every $X \geq P(T)$, $|X| \geq 3$ and the induced tournament $T(X)$ is critically CCF-indecomposable.*

The strongly connected case is more difficult to obtain. We have:

Proposition 9. *Given a strongly connected tournament T, with $|V(T)| \geq 3$, T is critically CCF-indecomposable if and only if either T is isomorphic to $\overrightarrow{C_k}$, where $|V(T)| = 2k+1$, or for each $X \in P(T)$, we have $|X| \geq 3$ and $T(X)$ is critically CCF-indecomposable.*

We now denote by $P_1(T) = \{X \in P(T), |X| = 1\}$. Propositions 8 and 9 lead us to associate with each critically CCF-indecomposable tournament T, such that $|V(T)| \geq 3$, the family $p(T)$ of the strong intervals X of T satisfying: $|X| \geq 2$ and $P_1(T(X)) \neq \emptyset$. It follows from Proposition 8 that for every $X \in p(T)$, $T(X)$ is strongly connected because $P_1(T(X)) \neq \emptyset$. Now, by Proposition 9, we obtain that $T(X)$ is isomorphic to $\overrightarrow{C_k}$, where $|X| = 2k + 1$. Consequently, $p(T)$ constitutes an interval partition of T and Theorem 2 follows.

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